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## Structure and commutativity of rings with constraints on nilpotent elements

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## STRUCTURE AND COMMUTATIVITY OF RINGS WITH CONSTRAINTS ON NILPOTENT ELEMENTS

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Recently [3] the authors proved the following :

**Theorem 1.** *Let  $R$  be an associative ring with a left identity, and let  $N$  be the set of nilpotent elements in  $R$ . Suppose that (P) for every  $x$  in  $R$ , there exists a positive integer  $n = n(x)$  and an element  $x'$  in the subring,  $\langle x \rangle$ , generated by  $x$  such that  $x^n = x^{n-1}x'$ , and (M)  $x - y \in N$  implies  $x^2 = y^2$  or both  $x$  and  $y$  commute with all elements of  $N$ . Then  $R$  is a subdirect sum of local commutative rings and nil commutative rings.*

As was noted in the final section of [3], Theorem 1 need *not* be true if the prime 2 in hypothesis (M) is replaced by *any prime*  $q > 2$ , even if we were to replace hypothesis (M) by the stronger hypothesis:  $x - y \in N$  implies  $x^2 = y^2$ . This naturally gives rise to the following question: What hypotheses, in addition to hypothesis (P) and the hypothesis that  $x - y \in N$  implies  $x^q = y^q$ , should be added in order to guarantee the commutativity of the ground ring  $R$ ? It turns out that one such hypothesis is to assume that the set  $N$  is commutative. In fact, by assuming this additional hypothesis on  $N$ , it turns out that *we do not need to assume that the ground ring  $R$  has a left identity* — an assumption which is absolutely essential for Theorem 1 to be true (see [3]). Our theorem, then may be stated as follows :

**Theorem 2.** *Let  $R$  be an associative ring and let  $N$  be the set of nilpotent elements of  $R$ . Suppose  $q$  is a fixed prime. Suppose, further, that (i)  $N$  is commutative, (ii) for every  $x$  in  $R$  there exists an element  $x'$  in  $\langle x \rangle$  and a positive integer  $n = n(x)$  such that  $x^n = x^{n-1}x'$ , (iii)  $x - y \in N$  implies  $x^q = y^q$ . Then  $R$  is a subdirect sum of local commutative rings and nil commutative rings.*

In preparation for the proof of Theorem 2 we first establish the following lemmas.

**Lemma 1.** *Let  $R$ ,  $N$ ,  $q$  be as in Theorem 2. Then,*

(a) *Hypothesis (ii) of Theorem 2 is equivalent to the following: For every  $x \in R$ , there exists an element  $x'$  in  $\langle x \rangle$  such that  $x - x^2x' \in N$ .*

(b) Hypothesis (iii) of Theorem 2 implies that

$$ab^q = b^q a, \text{ for all } a \in N \text{ and all } b \in R,$$

and necessarily all of the idempotent elements of  $R$  are in the center of  $R$ .

*Proof.* (a) is almost trivial. We shall prove (b). Since  $(a + b) - b \in N$ , therefore by hypothesis (iii),  $(a + b)^q = b^q$ . Hence,

$$b^q(a + b) = (a + b)^q(a + b) = (a + b)(a + b)^q = (a + b)b^q,$$

which simplifies to  $b^q a = ab^q$ . As is well known, every idempotent element commuting with all nilpotent elements is central.

**Lemma 2.** In the notation, and under the hypotheses, of Theorem 2, we have

(a)  $N$  is a commutative nil ideal of  $R$ .

(b) If  $f$  is a homomorphism of  $R$  onto  $R^*$ , then  $f(N)$  coincides with the set of all nilpotent elements of  $R^*$ .

*Proof.* The proof is obvious from the proof of Lemma [1]. However, for the sake of selfcontainedness, we shall give here the proof.

(a): Let  $a$  be an arbitrary element of  $N$ , and let  $b \in R$ . Suppose  $a^h = 0$ . By hypothesis (ii), there exists a  $c \in \langle ab \rangle$  such that  $(ab)^n = (ab)^{n+1}c$ . Let  $e = (ab)^n d$ , where  $d = c^n$ . Then, as is readily verified,

$$(1) \quad (ab)^n = (ab)^n e \text{ and } e^2 = e.$$

By Lemma 1 (b),  $e$  is in the center of  $R$ , and hence

$$e = e^2 = e(ab)^n d = aeb(ab)^{n-1}d = a \cdot e \{b(ab)^{n-1}d\}^h = 0,$$

since  $a^h = 0$ . Thus,  $e = 0$  and hence by (1),  $(ab)^n = 0$ . Therefore,  $ab$  is nilpotent. Similarly,  $ba$  is nilpotent. We have thus shown that  $ab$  and  $ba$  are nilpotent, for all  $a \in N$  and  $b \in R$ . Combining this and hypothesis (i), we conclude that  $N$  is a commutative nil ideal of  $R$ .

(b): Let  $d^*$  be an arbitrary nilpotent element of  $R^*$ , and let  $(d^*)^k = 0$ . Choose  $d$  in  $R$  such that  $f(d) = d^*$ . By Lemma 1 (a), there exists  $d' \in \langle d \rangle$  such that  $d - d^2 d' \in N$ . Since  $N$  is an ideal [by part (a)], we obtain

$$d - d^{k+1}(d')^k = (d - d^2 d') + dd'(d - d^2 d') + \dots + (dd')^{k-1}(d - d^2 d') \in N.$$

Recall that  $f(d) = d^*$ ,  $(d^*)^k = 0$ , and hence it follows that  $d^* \in f(N)$ . This proves part (b).

**Corollary 1.** *If  $R$  satisfies the hypotheses (i), (ii), (iii) of Theorem 2, then any subring of  $R$  and any homomorphic image of  $R$  satisfy (i), (ii), (iii).*

*Proof.* The statement is obvious for subrings. Now, let  $f$  be a homomorphism of  $R$  onto  $R^*$ . By Lemma 2 (b), it follows at once that  $R^*$  satisfies (i). Clearly,  $R^*$  satisfies (ii). To prove that  $R^*$  satisfies (iii), suppose that  $x^* - y^* \in f(N)$  [= set of nilpotents of  $R^*$ , by Lemma 2 (b)]. Then  $x^* - y^* = n^* = f(n)$ , for some  $n$  in  $N$ . Let  $x^* = f(x)$ ,  $y^* = f(y)$ . Since  $x - (x - n) \in N$ ,  $x^q = (x - n)^q$ , by (iii), and hence  $(x^*)^q = (x^* - n^*)^q = (y^*)^q$ . Thus,  $R^*$  satisfies (iii).

We are now in a position to prove Theorem 2.

*Proof of Theorem 2.* Since, as is well known, the ground ring  $R$  is isomorphic to a subdirect sum of subdirectly irreducible rings, we may in view of Corollary 1 assume that  $R$  is subdirectly irreducible. Recall that, by hypothesis (ii),  $x^n = x^{n+1}x'$ , and hence, as is readily verified,  $e = x^n(x')^n$  is idempotent and  $x^n = x^n e$ . Also, by Lemma 1 (b),  $e$  is a central idempotent in  $R$ , and hence  $e = 0$  or  $e = 1$  (if  $1 \in R$ ). Now, if  $R$  does not have an identity, then  $x^n = x^n e = 0$ , and hence  $R = N$  is commutative, by hypothesis (i). Next, suppose  $1 \in R$ . In this case,  $e = 0$  or  $e = 1$  implies that, for every  $x$  in  $R$ ,  $x$  is nilpotent or  $x$  is a unit in  $R$ ; that is,  $R$  is a local ring. To prove that  $R$  is commutative, we may further assume that  $R$  is finitely generated. By Lemma 1 (a) and Corollary 3.5 of [4], it readily follows that, for every  $x$  in  $R$ , there exists a positive integer  $n$  such that  $x - x^{n+1} \in N$ . Hence, by Jacobson's Theorem (see, e.g., [2]),  $R/N$  is a field. In fact, since  $R$  is finitely generated,  $R/N$  is a finite field, say,

$$(2) \quad R/N = \text{GF}(r), \text{ where } r = p^\alpha, \ p \text{ prime, } \alpha \geq 1.$$

Clearly, if  $N = \{0\}$ , then  $R$  is commutative. So, suppose  $N \neq 0$ , and let  $a$  be an arbitrary element of  $N$ . Suppose that  $a^k = 0$  but  $a^{k-1} \neq 0$ , ( $k \geq 2$ ). Since by hypothesis (iii),  $(1 + a)^q = 1$ , we have

$$(3) \quad qa^{k-1} = \{(1 + a)^q - 1\}a^{k-2} = 0.$$

On the other hand, by (2), the characteristic of  $R$  is equal to  $p^\beta$  for some positive integer  $\beta$ , and hence  $p^\beta a^{k-1} = 0$ . Combining this with (3), we get  $qa^{k-1} = 0 = p^\beta a^{k-1}$ . Now, if  $p \neq q$ , then the last implies that  $a^{k-1} = 0$ , which is a contradiction. This contradiction shows that  $p = q$ . Hence, by Lemma 1 (b), we have

$$(4) \quad ab' = b'a, \quad (a \in N, b \in R).$$

Moreover, by (2),  $b' - b \in N$ , and hence by hypothesis (i), we conclude that

$$(5) \quad a(b' - b) = (b' - b)a, \quad (a \in N, b \in R).$$

Combining (4) and (5), we see that

$$(6) \quad ab = ba, \quad (a \in N, b \in R).$$

Also, by (2), the multiplicative group of nonzero elements of  $R/N$  is cyclic. Combining this fact with (6), we can easily see that  $R$  is commutative. Note that in the above proof, we have shown that the ground ring  $R$  has the structure described in Theorem 2. This completes the proof.

It is well known that Theorem 2 need not be true if any of the hypotheses (i), (ii) is deleted (see [3]). In the subring  $R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$  of  $(\text{GF}(2))_2$ , all the hypotheses of Theorem 2 hold except (iii). Observe that  $R$  is not commutative, and hence hypothesis (iii) cannot be deleted. We conclude this paper with the following

**Remark.** Let  $q = 3$ . Let  $R$  be an algebra over  $\text{GF}(2)$  of dimension 4 with  $\{1, \xi, \eta, \eta'\}$  as a basis, and with the following multiplication table :

$$\begin{aligned} \xi\eta &= \eta', & \xi\eta' &= \eta + \eta' = \eta\xi, & \eta'\xi &= \eta, \\ \xi^2 &= 1 + \xi, & \eta\eta' &= \eta'\eta = \eta^2 = (\eta')^2 = 0. \end{aligned}$$

It can be verified that  $(R, +, \times)$  is an associative ring, and that all the hypotheses of Theorem 2 hold *except* that hypothesis (iii) is now replaced by that  $x - y \in N$  implies  $x^q = y^q$  or both  $x$  and  $y$  commute with all elements of  $N$  (where  $q = 3$ ). In verifying this, observe that, for  $x$  in  $R$ ,  $x^2 = x^3$ . Note, however, that  $R$  is not commutative. This example shows, then, that Theorem 2 need not be true if we replace hypothesis (iii) by the weaker hypothesis that  $x - y \in N$  implies  $x^q = y^q$  or both  $x$  and  $y$  commute with all elements of  $N$ .

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